JOURNAL OF COMPUTATIONAL PHYSICS 9, 194–206 (1972)

A Method for Determining the Eigenfrequency of a Mechanical System by Simulation

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Received February 17, 1971

A numerical method to determine the eigenfrequency of a mechanical system, using the finite-difference approximation and Fourier analysis, is presented. This method is powerful for the cases in which the medium is heterogeneous and the boundary is complicated. This is also advantageous when the degree of freedom of the system is very large. As examples, several cases of string and membrane vibration are studied.

INTRODUCTION

The problem of determining the eigenfrequency of a mechanical system has been studied for a long time and various methods have been presented by many authors. Of those methods, the semiexperimental procedure, in which the elastic vibration is experimentally excited by using supersonic technique and then the result is numerically analyzed by means of Fourier analysis, is adopted for the actual study of the elastic vibration problem. This method, however, is not necessarily semi-experimental, but we can work exclusively numerically by use of simulation. We can numerically calculate the disturbance which was caused by the excitation applied at any one of several points in the system, and then analyze these disturbances again numerically.

String Vibration

Homogeneous String

As an example a fundamental problem of string vibration is first studied. The differential equation of motion is

$$\nabla^2 u = (1/c^2)(\partial^2 u/\partial t^2) \tag{1}$$

in which $\nabla^2 u = \partial^2 u/\partial x^2$ and c is the propagation velocity.

This form of equation is modified to the following difference equation

$$u(t + \Delta t; x) = 2u(t; x) - u(t - \Delta t; x) + (c \cdot \Delta t)^2 \cdot \nabla^2 u, \tag{2}$$

where

$$\nabla^2 u = \{u(t; x + \Delta x) - 2 \cdot u(t; x) + u(t; x - \Delta x)\}/(\Delta x)^2$$

with which we can proceed one step in the time domain and calculate the disturbance at time $t + \Delta t$ using the data at $t - \Delta t$ and t.

The initial condition at $t = 0$ is given as

$$[u(t; x)]_{t=0} = U(x), \qquad [(\partial/\partial t)\, u(t; x)]_{t=0} = 0 \tag{3}$$

and the disturbance at $t = \Delta t$ is calculated not by Eq. (2) but by the following formula

$$\begin{aligned} u(\Delta t; x) &= U(x) + \frac{1}{2} \cdot \left[\frac{\partial^2 u}{\partial t^2}\right]_{t=0} (\Delta t)^2 \\ &= U(x) + \frac{1}{2} \cdot \left(\frac{c\Delta t}{\Delta x}\right)^2 [U(x + \Delta x) - 2U(x) + 2U(x - \Delta x)]. \end{aligned} \tag{4}$$

On the other hand the displacement $u(t; x)$ thus obtained can be expressed using the eigenfunction $e_k(x)$ and the eigenfrequency p_k in the following way

$$u(t; x) = \sum_k A_k e_k(x) \cdot \exp(ip_k t). \tag{5}$$

Consequently, by the Fourier analysis of $u(t; x)$, p_k is obtained as the peak frequency of the spectrum, and the spectrum amplitude

$$A(x, p_k) = \int_{-\infty}^{\infty} u(t; x) \cdot \exp(-ip_k t) \cdot dt$$

is nothing but $e_k(x)$ except a constant factor. Table I-A and Fig. 1 give the result

of numerical calculation in which the following values are employed:

Length of the string = 1.0,
Propagation velocity = c = 1.0,
Interval of $x = \Delta x = 1/15$,
Interval of $t = \Delta t = 1/20$.

TABLE I-A

Eigenvalues p_k of a Uniform String in Degree. Theoretical Values Show $n\pi$.
Vibration of a uniform string

(Angular velocity in degree)		
Mode	Theory	Numerical
1st	180	179.8
2nd	360	357.7
3rd	540	532.4
4th	720	702.0
5th	900	864.8
6th	1080	1019
7th	1260	1163
8th	1440	1296
9th	1620	1414
10th	1800	1605
11th	1980	
$x = 0(0.06667)\ 1.0$		

The time step is taken a little smaller than the critical value determined by the stability condition throughout the following calculations.

$u(t; x)$ is calculated for all the points $x = 1/15, 2/15, \ldots, 14/15$ for the time range $t = 0$–40.0. The eigenfrequency given by the Fourier analysis has good precision as shown in Table I-A. For the fundamental mode the error is only about 0.1 %, though for the higher modes it increases gradually.

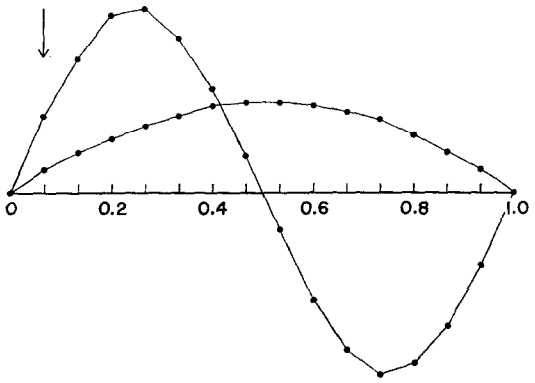

FIG. 1. Fundamental mode and the first higher mode of a string vibration, showing $\sin \pi x$ and $\sin 2\pi x$. Arrow indicates the point of excitation where the initial displacement was given. Initially no displacement was given at the other points.

This increase of error helps us to estimate the dependence of the accuracy on Δx. The error, when Δx is doubled or tripled, will be nearly equal to that for the 2nd and 3rd mode, respectively. The result, however, scarcely depends on the initial displacement $U(x)$, and the answers come out identically except for one or two units in the last digit for different $U(x)$. The form of $U(x)$, which takes a value different from zero only at a point next to the fixed end, was used because of an advantage that all the eigenfrequencies are obtained by a single calculation. The form of eigenfunction determined by spectrum amplitude is illustrated in Fig. 1, which shows sine curves, as expected theoretically, with slight irregularity. The time range was changed and the calculation repeated. As is shown in the following table the result came out amazingly constant in spite of the wide variety of time ranges.

Time range		0–10	0–20	0–30	0–40	0–50	0–60
Angular velocity	(1st mode)	179.6°	179.8°	179.8°	179.7°	179.8°	179.8°
	(2nd mode)	357.6°	357.7°	357.7°	357.7°	357.7°	357.8°

Heterogeneous String

Even if the string is not uniform the numerical procedure undergoes a change but little. As an example

$$c^2 = c^2(x) = a + b \cdot \cos(2\pi x) \qquad (0 \leqslant x \leqslant 1) \tag{6}$$

was assumed and the numerical solution was tried for the values $a = 1.0$, $b = 0.1$. For the fundamental mode

$$p = 1.023 \qquad (= 58.8°)$$

is given by the present method, which value, however, is also calculated from the approximation formula of Mathieu function [1] with two parameters a and b, and

$$p = 1.026$$

is obtained. This fact suggests that the eigenvalue of Mathieu function could be found from the present method of simulation.

TABLE I-B

Eigenvalues of a Square Membrane in Degree. Theoretical Values Correspond to

$$\frac{180}{2\pi} \cdot \sqrt{(n^2 + m^2)}\ (n, m = 1, 2, 3...)$$

Vibration of a square membrane
Size = 6.2832 × 6.2832 (Division 12 × 12)

		(Angular velocity in degree)	
Mode		Theory	Numerical
0	0	40.51	40.6
	1	64.06	63.6
	2	90.59	89.3
	3	118.12	114.8
	4	146.08	144.0
1	1	81.03	80.4
	2	103.29	102.3
	3	128.12	125.6
	4	154.28	148.5
2	2	121.54	120.2
	3	143.24	141.2
	4	167.05	162.6
3	3	161.82	159.7

RECTANGULAR MEMBRANE

Similar procedure can be applied to the two-dimensional problem. The fundamental equation is not different from Eq. (1) except that

$$\begin{aligned}\nabla^2 u &= (\partial^2/\partial x^2 + \partial^2/\partial y^2) \cdot u \\ &= (u(t; x + \Delta x, y) - 2u(t; x, y) + u(t; x - \Delta x, y))/(\Delta x)^2 \\ &\quad + (u(t; x, y + \Delta y) - 2u(t; x, y) + u(t; x, y - \Delta y))/(\Delta y)^2. \end{aligned} \tag{7}$$

TABLE II-A

Eigenvalues p_k of a Circular Membrane in Degree. Theoretical Values Correspond to $180/\pi \cdot p_{0k}$, Where p_{0k} Is the k-th Root of $J_0(p) = 0$.[a]
Vibration of a circular membrane

		(Angular velocity in degree)		
Mode		Theory	Numerical	
Radial	Azimuthal		Polar coord.	Rectangular coord.
0	0	137.8	137.8	137.0
	1	219.6	218.6	216.5
	2	294.3	289.7	290
	3	365.5	344.3	—
1	0	316.3	314.7	306
	1	402.0	398.6	382
	2	482.3	474.0	455
2	0	495.5	489.7	—
3	0	675.6	660.5	—
			division (R-direction) .LE. 14	$DX = DY = R/4$

[a] Mesh sizes are chosen from the values $\Delta R = \frac{1}{8}\text{–}\frac{1}{14}$ $\Delta\theta = 10°\text{–}30°$, depending on the number of nodes.

As a simple and basic problem a square membrane with all the edges fixed is studied. The initial condition is given with no essential difference from the one-dimensional problem. The answer for a square (size $= 2\pi \times 2\pi$, divided into 12×12; the initial displacement assumed at $x = 2\pi \cdot (2/12)$, $y = 2\pi \cdot (4/12)$ and no other points) is shown in Table I-B together with the exact theoretical answer.

Circular Membrane

Circular membranes can be treated similarly to the previous problem. In this case polar coordinates are conveniently used and the formula

$$\nabla^2 u = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + \frac{1}{r^2} \cdot \frac{\partial^2}{\partial \theta^2}\right) \cdot u \tag{8}$$

is employed instead of Eq. (7). [For the center of the circle, the formula referred to rectangular coordinates $\nabla^2 u = (\partial^2/\partial x^2 + \partial^2/\partial y^2) \cdot u$ is used to avoid the difficulty that comes from $r = 0$.] The rectangular coordinates, however, are also usable. In this case the boundary condition is given using the formula for an irregular star based on the linear interpolation [2].

Table II-A gives the result of both calculations. The last column based on the rectangular coordinates has a precision poorer than the other, but this is a natural consequence if we remember the coarse interval employed in this solution, and the form of eigenfunctions that came out as the spectral amplitude is adequate showing the distribution of Bessel function $J_0(x)$ in Fig. 2.

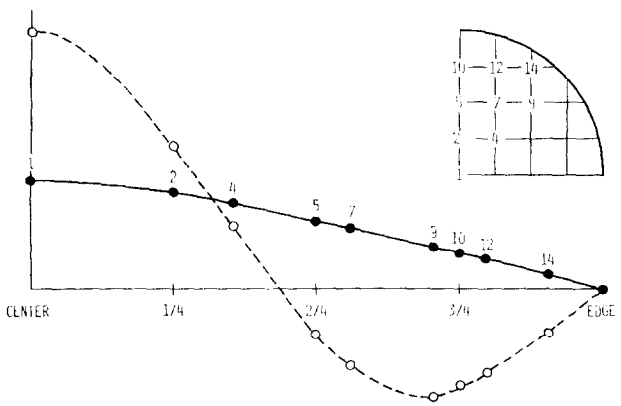

FIG. 2. Fundamental mode (●) and the first higher mode (○) of circular membrane. Numbers on curves correspond to the points in the quadrant. The initial displacement is assumed at the point 2.

MEMBRANE BOUNDED BY CONCENTRIC CIRCLES

Two concentric circular edges $r = r_1$ and $r = r_2$ add no difficulty to the case of a simple circular membrane. The solution was obtained by two methods, first using the formula (8) and second transforming Eq. (8) by the formula $\rho = \log r$ into

$$\nabla^2 u = \exp(-2\rho) \cdot \left(\frac{\partial^2}{\partial \rho^2} + \frac{\partial^2}{\partial \theta^2}\right) \cdot u. \tag{9}$$

No distinction can be found in the precision between two methods, and the answer is given in Table II-B.

As is seen in this table, the frequencies in the 1st (0–0 mode) and 2nd (0–1 mode, one diameter as a mode) lines differ but little, and if r_1 approaches r_2 this difference will become even smaller. This is a situation that causes difficulty to numerical methods. In such a case, however, the present method works usefully, because by assuming the initial displacement with a circular symmetry the asymmetric modes are suppressed, while by the asymmetric initial displacement, the fundamental mode is removed, and thus the neighbouring frequencies are easily obtained separately.

TABLE II-B

Eigenvalues p_k of a Circular Membrane with a Core in Degree.
Vibration of a circular membrane with a core
($R_1 = 0.5$, $R_2 = 1.0$)

Mode		(Angular velocity in degree)	
Radial	Azimuthal	Theory	Numerical
0	0	357.9	357.2
	1	366.3	365.6
	2	390.4	389.0
1	0	718.9	715.4
	1	723.4	719.7
2	0	1079.2	1059.8
			number of division in R-direction = NR .LE. 14

Similarly, if the points are picked up which are on the nodal line of a certain mode, the spectra at those points lack the line corresponding to that mode, and in this way the resolving power of the analysis is increased. In the present example, which is not very much ill-natured, all the frequencies could be obtained by the initial displacement which is all zero except at a single point, if the time range is taken long. The above technique, however, was successfully tested in this case with improvement of accuracy and saving of computer time.

Membrane Bounded by Eccentric Two Circles

The eigenvibrations of a membrane bounded by eccentric two circles were calculated by one of the authors using the perturbation method for the case of small eccentricity [3]. A similar procedure as the previous section, however, can be applied for this kind of problem, too. By the use of bipolar coordinates ξ and η, which are connected with the original coordinates x and y by the transformation formula

$$\begin{aligned} x &= a \cdot \sinh \xi/(\cosh \xi + \cos \theta), \\ y &= a \cdot \sin \theta/(\cosh \xi + \cos \theta), \end{aligned} \tag{10}$$

the Laplacian is given as [4]

$$\nabla^2 u = \left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \theta^2}\right) u \cdot \left(\frac{\cosh \xi + \cos \theta}{a}\right)^2, \tag{11}$$

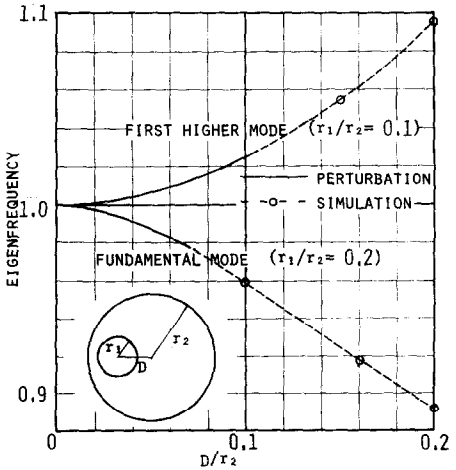

FIG. 3. Variation of the eigenvalues of the fundamental and the first higher modes for a membrane bounded by eccentric two circles. Fundamental mode for $D = 0$ (concentric circles) is the root of $J_0(kr_1) \cdot N_0(kr_2) - N_0(kr_1) \cdot J_0(kr_2) = 0$ and the first higher mode for $D = 0$ is the root of $J_1(kr_1) \cdot N_1(kr_2) - N_1(kr_1) \cdot J_1(kr_2) = 0$.

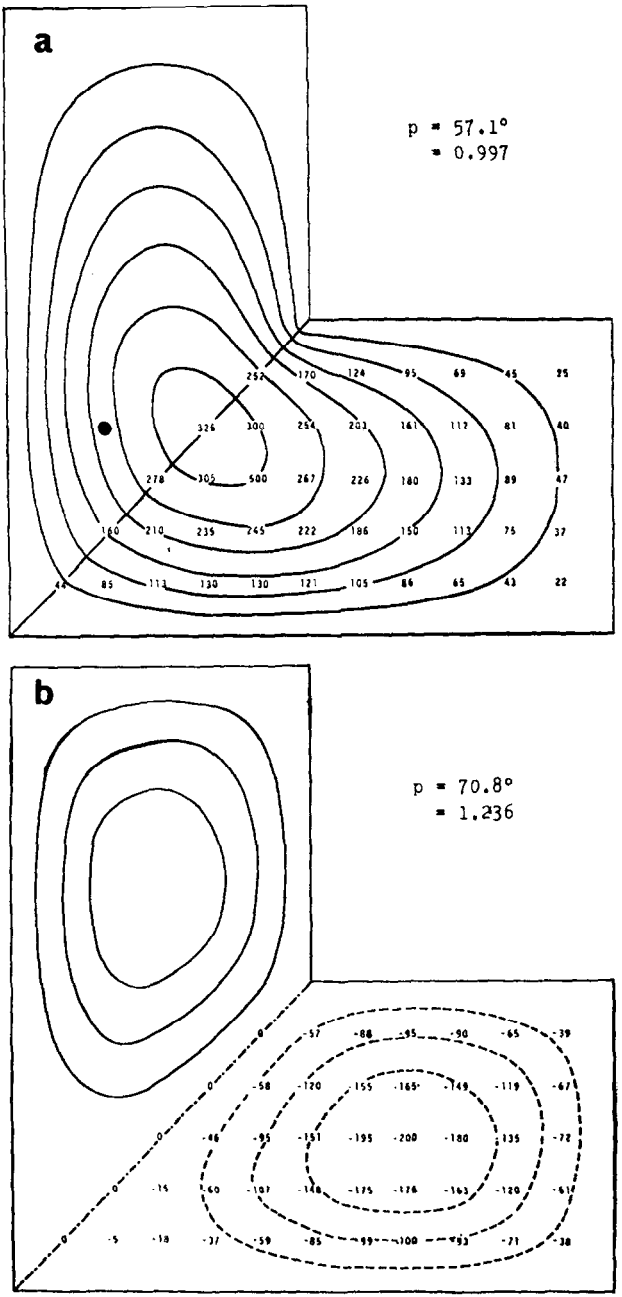

FIG. 4(a–b). Oscillation of an *L*-shape membrane. The numerals, which are proportional to the displacement, and contour lines show the form of oscillation. Black circle is the point of excitation where the initial displacement was given. No displacement was given to the other points. Including this initial condition all the parameters for computation are same as the problem of square membrane. (a) Fundamental mode. (b) First higher mode of an *L*-shape membrane, solid and broken lines show positive and negative displacements respectively. Chain line is the nodal line.

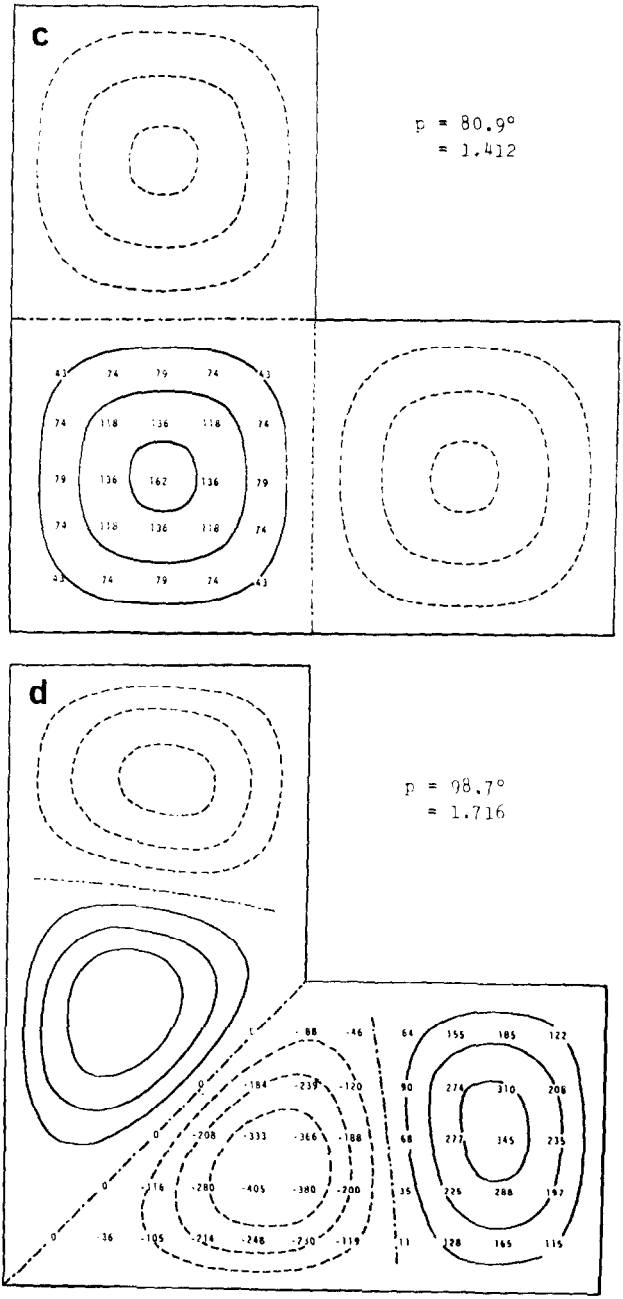

FIG. 4(c–d). (c) Second higher mode; mode of oscillation of every one of three squares is the same, showing the fundamental mode of a single independent square membrane. (d) Third higher mode.

where the parameter a is

$$a^2 = (r_2^2 - r_1^2 - D^2)^2/4D^2 - r_1^2. \tag{12}$$

r_1, r_2 are the radii of inner and outer circles, respectively, and D is the distance between the two centers. The equations of the two circles referred to new coordinates are given by

$$\xi_{1,2} = \log\{(a/r_{1,2}) + \sqrt{[(a/r_{1,2})^2 + 1]}\}. \tag{13}$$

Numerical results for the eccentricity not so small as the previous study are shown in Fig. 3, where the solid lines are the results obtained by the perturbation method and the circles are the values obtained by the present method.

The technique explained in the previous example is also used in this case. $\Delta\xi = 1/15$, $\Delta\theta = 15°$ are used in most of the calculations for this example.

L-Shape Membrane

As the example of a problem with a special form of boundaries for which the present method has an advantage, the oscillation of an L-shape membrane was solved. Except the boundary form, which is an L-shape, a quarter of a square removed, the equations are not different from those in Eq. (7). The initial condition, the grid spacing and the time step are the same as the problem of a square.

The answer is obtained for the first four modes and is given in Fig. 4. The numerals, which are proportional to the displacement, and contour lines show the form of oscillation.

The second higher mode [Fig. 4(c)], in which the membrane is divided into three identical squares, shows a feature like the oscillation of a square membrane with two crossing nodal lines, and suggests the legitimacy of the present treatment. The eigenfrequencies of both cases lie within a small computational error and the eigenfunction given by the numerals in an individual square is as expected, showing a feature of the fundamental mode of a square.

Discussion and Conclusion

The present method based on the numerical simulation will be of use for the determination of eigenfrequencies as well as eigenfunctions of mechanical systems such as string, membrane, beam, plate and so on.

The Rayleigh–Ritz method, useful in many problems, requires an approximate form of eigenfunctions in advance. This is, however, not always easy when the

boundary shape is complicated and the medium is heterogeneous. The difficulty increases for higher modes, for which the approximate form of eigenfunctions is not easy to obtain. Actually, in the study of the eigenfrequency of the earth, this difficulty proved to be a big barrier [5]. In such a case the present method is powerful. It can be used without any knowledge of the form of eigenfunction, and if the knowledge about the nodal lines and the symmetric property of modes are added, the resolving power is increased by an appropriate choice of the initial displacement. In this way neighboring spectral lines are obtained separately by the removal of disturbing modes, and besides, the eigenfunction is obtained as the spectral amplitude together with the frequency as the result of Fourier analysis. This is also a desirable property of this method.

Compared with the method in which all the displacement components are calculated as the roots of linear algebraic equations, this is not an economical way of computation when the degree of freedom of the system, N, is not large. For one-dimensional problems or two-dimensional problems with simple boundary forms the value of N is at most several hundreds, and matrix calculation will not be difficult. However, the problems with complicated boundaries or strongly heterogeneous media require a fine grid spacing, and N easily becomes 1,000 or more. In the problems of elastic solids, in which according to the dimensions of the space, two or more unknowns have to be assumed for one point, N often becomes as large as 10,000. In such a case the situation changes. The handling of so many unknowns, though feasible in principle, would clearly make heavy demands on computer time, expense and technique. If the present method is employed for the above size of problem, no special technique is necessary, and a middle size computer (say, 32K word available core memory) is enough, the computer time being only proportional to N, and neither to N^2 nor to N^3.

The numerical computation was carried out by UNIVAC 1108 through the project UNICON.

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